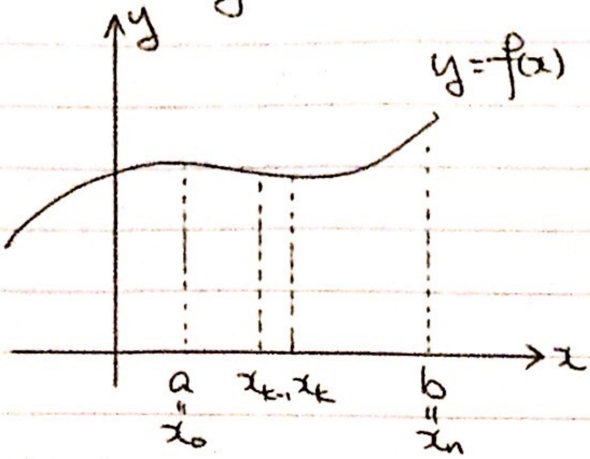
Definite Integration :



Area under $f(x)$ over $[a,b] \approx \sum_{i=1}^{n} f(x_i)\,\Delta x_i$ $\qquad \Delta x_i = x_i - x_{i-1}$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i$$

Fundamental Theorem of Calculus :

If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function,
let $F(x) = \int_{x_0}^{x} f(t)\,dt$, then $F'(x) = f(x)$.

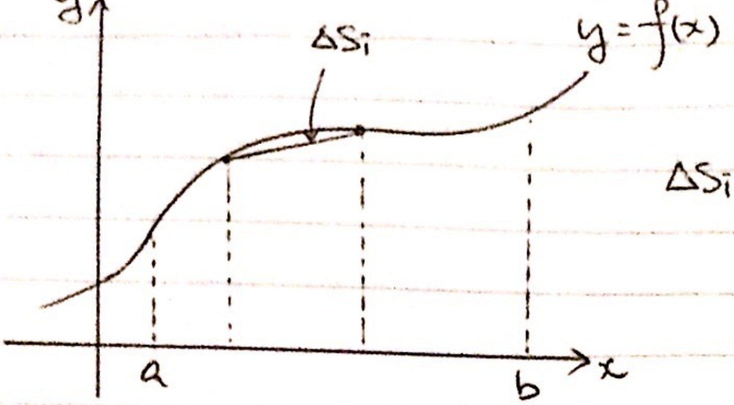
Direct consequence : $F(b) - F(a) = \int_{x_0}^{b} f(x)\,dx - \int_{x_0}^{a} f(x)\,dx$

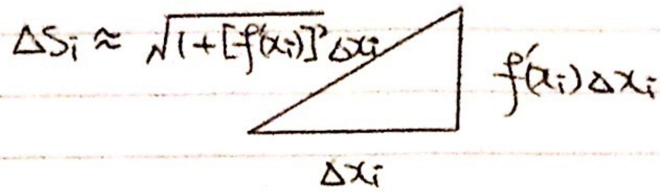
$$= \int_a^b f(x)\,dx$$

Idea : Computing definite integrals

" = " Finding infinite sums.

(Not necessary to be finding areas)

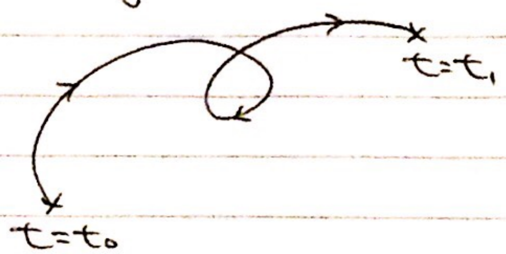




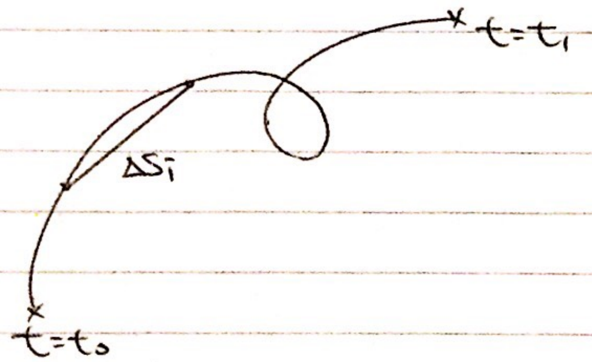
$$\text{Arclength} \approx \sum_{i=1}^{n} \Delta S_i \approx \sum_{i=1}^{n} \sqrt{1+[f'(x_i)]^2}\,\Delta x_i$$

$$\text{Arclength} = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+[f'(x_i)]^2}\,\Delta x_i = \int_a^b \sqrt{1+[f'(x)]^2}\,dx$$

Parametrized curve : $\gamma(t) = (x(t), y(t))$ $\quad t_0 \le t \le t_1$



e.g. $\gamma(t) = (x(t), y(t)) = (R\cos t, R\sin t)$, $0 \le t \le 2\pi$, $R>0$.
Circle centered at $(0,0)$ with radius R.



$\Delta S_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}$

$$\begin{aligned}\text{Arclength} &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta S_i \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2}\,\Delta t_i \\ &= \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt\end{aligned}$$

In particular, if $x = t$, $\gamma(x) = (x, y(x)) = (x, f(x))$

it reduces to $\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$.

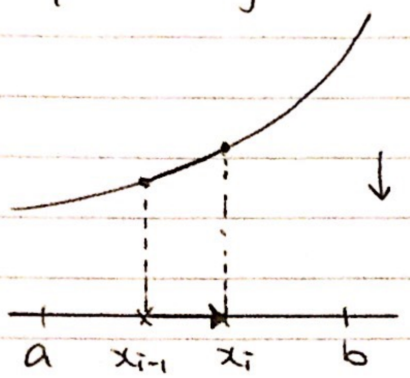
e.g. $(x(t), y(t)) = (R\cos t, R\sin t)$, $0 \le t \le 2\pi$, $R > 0$

$(x'(t), y'(t)) = (-R\sin t, R\cos t)$

$\sqrt{[x'(t)]^2 + [y'(t)]^2} = R$

Arclength $= \int_0^{2\pi} R\, dt = 2\pi R$.

Another point of view:



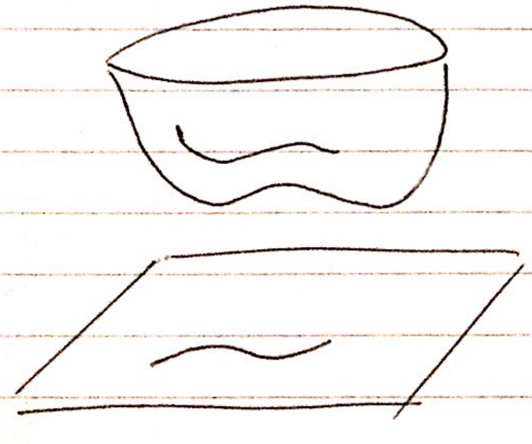
$\downarrow$ projection.

$\Delta s_i = \sqrt{1 + [f'(x_i)]^2}\, \Delta x_i$

$ds = \sqrt{1 + [f'(x)]^2}\, dx$

What we need:
As x increases, how long it travels.
(called metric).

Higher dimension: usual metric $ds^2 = dx^2 + dy^2$



In general: $ds^2 = ?$